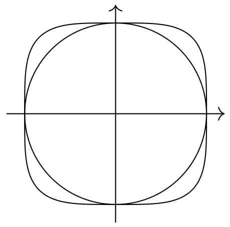
4401. (a) The suggested equations, when raised to the fourth, are $x^4 = \cos^2 t$ and $y^4 = \sin^2 t$. Adding these, $x^4 + y^4 = \cos^2 t + \sin^2 t \equiv 1$.

(b) If the inequality is well defined, $\cos t \in [0, 1]$. Squaring a number in $[0, 1]$ leaves it unchanged or decreases it. Hence, $\sqrt{\cos t} \geq \cos t$ always holds, if it is well defined.

(c) The inequality $\sqrt{\sin t} \geq \sin t$ also holds, if the square root is well defined. So, in the positive quadrant, the curve $x^4 + y^4 = 1$ is everywhere on or outside the unit circle, whose parametric equations are $x = \cos t$, $y = \sin t$. The curve is



4402. (a) Consider the following binomial expansion, in which $a \neq 0$:

$$a\left(x + \tfrac{b}{3a}\right)^3 \equiv ax^3 + bx^2 + \text{a linear function of } x.$$

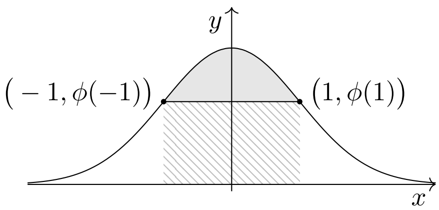
This matches the first two terms of the cubic. It must then be possible to write the linear function of x that remains as a linear function of $\left(x + \frac{b}{3a}\right)$. So, with $\alpha = \frac{b}{3a}$, we have the required form.

(b) The curve $C : y = px^3 + qx$ has rotational symmetry around O, because it consists solely of odd powers of x. The curve

$$D : y = p(x + \alpha)^3 + q(x + \alpha) + r$$

is then a translation of C by vector $-\alpha\mathbf{i} + r\mathbf{j}$. So, has rotational symmetry around the point $(-\alpha, r)$. Every cubic, therefore, has rotational symmetry. □

4403. The points of inflection of a normal distribution are at $x = \mu \pm \sigma$. In this case, for $X \sim N(0, 1)$, these are at $x = \pm 1$.



Using a calculator, the total shaded area is

$$\mathbb{P}(-1 < X < 1) = 0.68268...$$

The hatched rectangle has area $2\phi(1)$, which is 0.48394... So, the area enclosed is the difference between the two, which is 0.199 (3sf).

4404. The integrand is

$$\frac{2x^2}{x^2 - 16} \equiv \frac{2(x^2 - 16) + 32}{x^2 - 16} \equiv 2 + \frac{16}{x^2 - 16}.$$

For partial fractions,

$$\frac{32}{(x+4)(x-4)} \equiv \frac{A}{x+4} + \frac{B}{x-4}$$
$$\Longrightarrow 32 \equiv A(x-4) + B(x+4).$$

Substituting $x = \pm 4$, we get $A = -4$ and $B = 4$. So, the integral is

$$\int_5^8 2 - \frac{4}{x+4} + \frac{4}{x-4}\, dx$$
$$= \left[2x + 4\ln\left|\frac{x-4}{x+4}\right|\right]_5^8$$
$$= \left(16 + 4\ln\tfrac{4}{12}\right) - \left(10 + 4\ln\tfrac{1}{9}\right)$$
$$= 6 + 4\ln 3, \text{ as required.}$$

4405. (a) P has no points of inflection, so $\mathrm{f}''(x)$ does not change sign. Since f is a polynomial, $\mathrm{f}''(x)$ is always non-negative or non-positive. Wlog, we choose $\mathrm{f}''(x) \geq 0$ for all x. The proof is the same, *mutatis mutandis*, if $\mathrm{f}''(x) \leq 0$ for all x.

(b) Differentiating the definition of h,

$$\mathrm{h}(x) = \mathrm{f}(x) - mx - c$$
$$\Longrightarrow \mathrm{h}'(x) = \mathrm{f}'(x) - m$$
$$\Longrightarrow \mathrm{h}''(x) = \mathrm{f}''(x).$$

And $\mathrm{f}''(x) \geq 0$ for all x, so $\mathrm{h}''(x) \geq 0$ for all x.

(c) $\mathrm{h}(x) = 0$ holds at intersections of $y = \mathrm{f}(x)$ and $y = mx + c$. So, Q has at least three x intercepts. The first three are at $x = a, b, c$.

(d) Q has non-negative curvature, so any chord lies at or above Q. The x interval $[a, b]$, along the x axis, is a chord of Q, so $\mathrm{h}(x) \leq 0$ on $[a, b]$. And there are no x intercepts of Q between a and b (a, b, c are the *first* three x intercepts). So, $\mathrm{h}(x) < 0$ on (a, b).
The same argument applies for $[b, c]$.
Hence, $\mathrm{h}(x) < 0$ on $(a, b) \cup (b, c)$.

(e) We know that $\mathrm{h}(b) = 0$. But this, combined with the results of part (d), means that $(b, 0)$ is a local maximum. Hence, $\mathrm{h}''(b) < 0$. This is a contradiction.
So, no three points on P are collinear. QED.

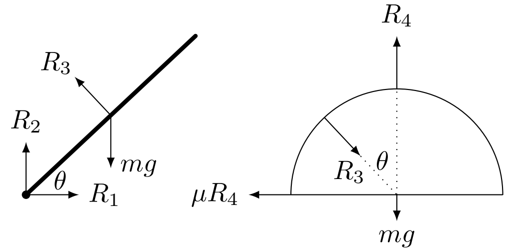
4406. This is not true. Consider a possibility space of eight equally likely outcomes $\{1, 2, 3, 4, 5, 6, 7, 8\}$. Define events X, Y, Z as follows:

$$X = \{1, 2, 3, 4\}, Y = \{2, 3, 4, 5\}, Z = \{3, 4, 5, 6\}.$$

Each has probability $\frac{1}{2}$. X and Y are dependent, as each increases the probability of the other to $\frac{3}{4}$. Likewise Y and Z. But X and Z are independent:

$$\mathbb{P}(X \mid Z) = \mathbb{P}(Z \mid X) = \tfrac{1}{2}.$$

4407. Consider the case of limiting friction, in which the objects are in equilibrium, with friction at F_{max}. The force diagrams are



Taking moments around the hinge, $R_3 = mg\cos\theta$. Resolving vertically for the block,

$$\begin{aligned} & R_4 - R_3\cos\theta - mg = 0 \\ \Longrightarrow\ & R_4 = mg\cos^2\theta + mg \\ & \equiv mg(\cos^2\theta + 1) \\ & \equiv \tfrac{1}{2}mg(\cos 2\theta + 3). \end{aligned}$$

Resolving horizontally for the block,

$$\begin{aligned} & R_3\sin\theta - \mu R_4 = 0 \\ \Longrightarrow\ & mg\sin\theta\cos\theta - \mu\tfrac{1}{2}mg(\cos 2\theta + 3) \\ \Longrightarrow\ & \mu = \frac{2\sin\theta\cos\theta}{\cos 2\theta + 3} \\ & \equiv \frac{\sin 2\theta}{\cos 2\theta + 3}. \end{aligned}$$

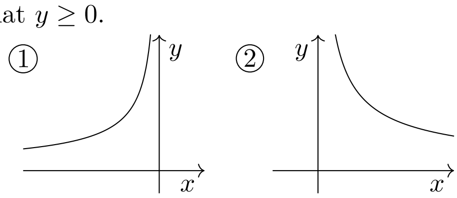
Therefore, for equilibrium,

$$\mu \geq \frac{\sin 2\theta}{\cos 2\theta + 3}.$$

4408. The first equation is a quadratic in $xy^{\frac{3}{2}}$:

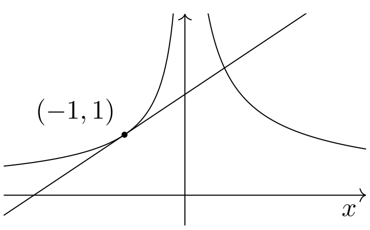
$$\begin{aligned} & \big(xy^{\frac{3}{2}} + 1\big)\big(xy^{\frac{3}{2}} - 2\big) = 0 \\ \Longrightarrow\ & xy^{\frac{3}{2}} = -1, 2. \end{aligned}$$

Consider each of these as a separate graph. The relationships are akin to inverse proportion, with an additional requirement due to the presence of $y^{\frac{1}{2}}$ that $y \geq 0$.

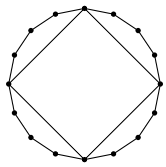


$xy^{\frac{3}{2}} = -1$ $\qquad$ $xy^{\frac{3}{2}} = 2$

We know that $3y = 2x + 5$ is tangent to the curve. Since it has +ve gradient, the point of tangency is in the second quadrant, with ①. It is at $(-1, 1)$. Putting the graphs back together, it follows that there must be one more point of intersection, with ②, in the positive quadrant:



4409. The scenario, with a successful outcome shown, is



Choose the first vertex without loss of generality. This fixes the set of positions for the remaining three vertices: the full set must have rotational symmetry order 4. So, the probability that the subsequent vertices are all selected from this set of three is

$$p = 1 \times \tfrac{3}{15} \times \tfrac{2}{14} \times \tfrac{1}{13} = \tfrac{1}{455}.$$

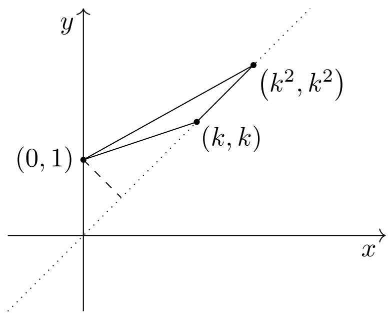
4410. Rearranging the second equation, $\tan t = \frac{2}{P}$ and $\cot t = \frac{P}{2}$. Using double angles on the first,

$$\begin{aligned} & 2P\sin t\cos t = 1 \\ \Longrightarrow\ & 4P^2\sin^2 t\cos^2 t = 1 \\ \Longrightarrow\ & 4P^2 = \operatorname{cosec}^2 t\sec^2 t \\ & \equiv (1 + \cot^2 t)(1 + \tan^2 t) \\ & = \left(1 + \tfrac{P}{2}^2\right)\left(1 + \tfrac{2}{P}^2\right) \\ & \equiv \frac{(P^2 + 4)^2}{4P^2} \\ \Longrightarrow\ & 16P^4 = (P^2 + 2)^2 \\ \Longrightarrow\ & P = \pm\tfrac{2}{\sqrt{3}}. \end{aligned}$$

The solution is

$$\begin{aligned} P &= \tfrac{2}{\sqrt{3}}, & t &= \tfrac{\pi}{3}, \tfrac{4\pi}{3}, \\ P &= -\tfrac{2}{\sqrt{3}}, & t &= \tfrac{2\pi}{3}, \tfrac{5\pi}{3}. \end{aligned}$$

4411. The vertices (k, k) and $\big(k^2, k^2\big)$ are both on $y = x$. So, to find the area of the triangle, we need only find the "height" of the triangle, dashed below, which is the distance of $(0, 1)$ from $y = x$.



Using $A_\triangle = \frac{1}{2}bh$,

$$\begin{aligned} A_\triangle &= \tfrac{1}{2} \cdot \sqrt{2}(k^2 - k) \cdot \tfrac{1}{2}\sqrt{2} \\ &= \tfrac{1}{2}k(k - 1), \text{ as required.} \end{aligned}$$

4412. Consider the quantity

$$S = x + y + z.$$

Initially, this is zero. A step in any direction adds or subtracts 1 from one of the coordinates (x, y, z), thereby adding or subtracting 1 from S. So, after one step S is odd, after two steps S is even, and so on. After six steps, S is even. But $(1, 1, 1)$ gives $S = 3$. Hence, the probability of ending up at $(1, 1, 1)$ after six steps is zero.

4413. (a) Integrating the velocity,

$$\begin{aligned} \mathrm{s}(\delta t) &= \int_0^{\delta t} \sqrt{2t+9}\, dt \\ &\equiv \left[\tfrac{1}{3}(2t+9)^{\frac{3}{2}} \right]_0^{\delta t} \\ &\equiv \tfrac{1}{3}(2\delta t + 9)^{\frac{3}{2}} - 9. \end{aligned}$$

(b) Using the binomial expansion,

$$\begin{aligned} &\tfrac{1}{3}(2\delta t + 9)^{\frac{3}{2}} - 9 \\ \equiv\ & 9\big(1 + \tfrac{2}{9}\delta t\big)^{\frac{3}{2}} - 9 \\ \approx\ & 9\big(1 + \tfrac{1}{3}\delta t + \tfrac{1}{54}\delta t^2\big) - 9 \\ \equiv\ & 3\delta t + \tfrac{1}{6}\delta t^2. \end{aligned}$$

(c) The particle begins with velocity $u = 3$. With constant acceleration a, we use $s = ut + \frac{1}{2}at^2$. The displacement is

$$\mathrm{s}(\delta t) = 3\delta t + \tfrac{1}{2}a\delta t^2.$$

With $a = \frac{1}{3}$, this is the same as the previous quadratic approximation, as required.

4414. Rotation by 90° clockwise around the origin maps (p, q) to $(q, -p)$. This is the same as a reflection in $y = 0$, mapping (p, q) to $(-p, q)$, then a reflection in $y = x$, mapping $(-p, q)$ to $(q, -p)$. We apply these in turn.

Firstly, reflecting in $y = 0$,

$$\mathrm{g}(x) = \mathrm{h}(y) \longmapsto \mathrm{g}(-x) = \mathrm{h}(y).$$

Then, reflecting in $y = x$,

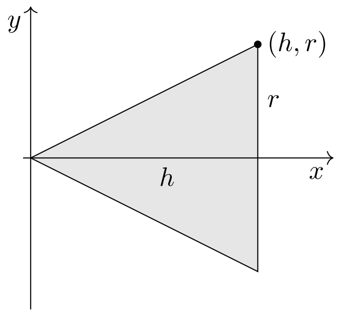
$$\mathrm{g}(-x) = \mathrm{h}(y) \longmapsto \mathrm{g}(-y) = \mathrm{h}(x).$$

So, the required equation is $\mathrm{g}(-y) = \mathrm{h}(x)$.

— Nota Bene —

The second transformation, switching x and y, should be considered carefully. It is not the input $-x$ of the function $\mathrm{g}(*)$ which is switched with the input y of the function $\mathrm{h}(*)$. Rather, it is the input x of the function $\mathrm{g}(-*)$ which is switched with the input of the function $\mathrm{h}(*)$.

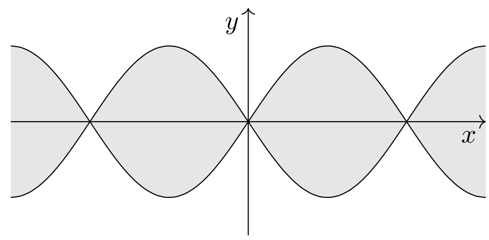
4415. We place the apex of the cone at the origin, and the base at $x = h$. In cross-section, the cone is



The equation of the upper slanted edge is $y = \frac{r}{h}x$, where y represents the radius of a thin disc. So, the volume of the cone is given by integrating πy^2 across the height of the cone:

$$\begin{aligned} V_{\text{cone}} &= \int_0^h \pi \left(\frac{rx}{h}\right)^2 dx \\ &\equiv \left[\pi \frac{r^2x^3}{3h^2}\right]_0^h \\ &\equiv \pi \frac{r^2h^3}{3h^2} \\ &\equiv \tfrac{1}{3}\pi r^2 h, \text{ as required.} \end{aligned}$$

4416. The boundary equations are $y = \pm \sin x$. Points in the solution set have y values closer to zero than points on these sinusoids. This gives



4417. (a) Writing over base 64,

$$\log_{64}(x+1)^6 + \log_{64} x^3 + \log_{64}(x+1)^2 = 0.$$

We combine the logs and exponentiate:

$$\begin{aligned} &\log_{64} x^3(x+1)^8 = 0 \\ \Longrightarrow\ & x^3(x+1)^8 = 1 \\ \Longrightarrow\ & x^3(x+1)^8 - 1 = 0. \end{aligned}$$

(b) At $x = 0$, the LHS has value -1; at $x = 1$, it has value 255. There are no discontinuities, so, due to the sign change, there is a root $x \in (0, 1)$.

The LHS of the original equation is undefined at $x = 0$, but tends to $-\infty$ as $x \to 0^+$. At $x = 1$, the LHS has value $\frac{4}{3}$. Again, there is a sign change, confirming that E has a root in the domain $(0, 1)$.

4418. Assuming limiting friction, friction is at maximum and neither block is moving. The tensions are therefore m_1g and m_2g, where $m_1g > m_2g$. The total downwards force on the pulley is the sum of the tensions, i.e. $R = m_1g + m_2g$. We are told that the difference between the tensions is equal to μR. This gives

$$\begin{aligned} & m_1g - m_2g = \mu(m_1g + m_2g) \\ \Longrightarrow\ & m_1 - m_2 = \mu m_1 + \mu m_2 \\ \Longrightarrow\ & m_1(1-\mu) = m_2(1+\mu) \\ \Longrightarrow\ & \frac{m_1}{m_2} = \frac{1+\mu}{1-\mu}, \text{ as required.} \end{aligned}$$

4419. Using the identity $\cos x \equiv \sin\left(\frac{\pi}{2} - x\right)$,

$$\mathrm{f}(x) = (1-k)\sin x + k\cos x.$$

When expressed in harmonic form, the amplitude of such a sum of sinusoids is the Pythagorean sum of the individual amplitudes. So,

$$\begin{aligned} A^2 &= (1-k)^2 + k^2 \\ &\equiv 2k^2 - 2k + 1. \end{aligned}$$

Taking the positive root by definition of A,

$$A = \sqrt{2k^2 - 2k + 1}, \text{ as required.}$$

4420. (a) Differentiating implicitly,

$$2xy + x^2\tfrac{dy}{dx} + 2 + \tfrac{dy}{dx} = 0.$$

Setting $\frac{dy}{dx} = 0$ gives $2xy + 2 = 0$. Substituting into the original equation,

$$\begin{aligned} & x^2 \cdot -\tfrac{1}{x} + 2x - \tfrac{1}{x} = 0 \\ \Longrightarrow\ & x = \pm 1. \end{aligned}$$

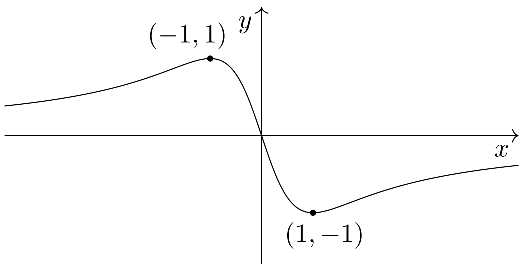
This gives SPs at $(\pm 1, \mp 1)$, as required.

(b) Rearranging to make y the subject,

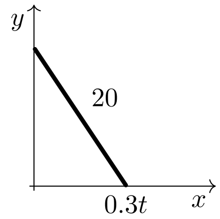
$$y = -\frac{x}{1+x^2}.$$

Since the denominator is a polynomial of greater degree than the numerator, $y \to 0$ as $x \to \pm\infty$. So, the x axis is an asymptote.

(c) The curve passes through the origin and has no other axis intercepts. It has odd symmetry. Putting these facts together with the answers from parts (a) and (b), the graph is



4421. The scenario, with the ground as the x axis and the wall as the y axis, is as follows. Assuming that the foot of the ladder is at the origin at time $t = 0$, the x coordinate of A is given by $x = 0.3t$.



Pythagoras gives the y coordinate of B as

$$\begin{aligned} y &= \sqrt{20^2 - (0.3t)^2} \\ &\equiv \sqrt{400 - 0.09t^2}. \end{aligned}$$

Differentiating with respect to time,

$$\frac{dy}{dt} = \tfrac{1}{2}\left(400 - 0.09t^2\right)^{-\frac{1}{2}}(-0.18t).$$

When B is 12 feet off the ground, the x coordinate of A is 16 feet, which gives $t = 160/3$. This gives

$$\left.\frac{dy}{dt}\right|_{t=160/3} = -0.4.$$

So, at this instant, end B is sliding down the wall at 0.4 feet per second, as required.

4422. In partial fractions, the integrand is

$$\frac{1}{16t - t^3} \equiv \frac{1}{16t} - \frac{1}{32(t+4)} - \frac{1}{32(t-4)}.$$

Hence, the integral is

$$\begin{aligned} & \int_1^2 \frac{1}{16t} - \frac{1}{32(t+4)} - \frac{1}{32(t-4)}\,dt \\ =\ & \left[\tfrac{1}{16}\ln|t| - \tfrac{1}{32}\ln|t+4| - \tfrac{1}{32}\ln|t-4|\right]_1^2 \\ =\ & \left(\tfrac{1}{16}\ln 2 - \tfrac{1}{32}\ln 6 - \tfrac{1}{32}\ln 2\right) \\ & \quad - \left(\tfrac{1}{16}\ln 1 - \tfrac{1}{32}\ln 5 - \tfrac{1}{32}\ln 3\right) \\ =\ & \tfrac{1}{32}(\ln 2 - \ln 6 + \ln 5 + \ln 3) \\ =\ & \tfrac{1}{32}\ln 5. \end{aligned}$$

4423. (a) Substituting into the LHS,

$$\begin{aligned} & \left(2y - \sqrt{3}x\right)^2 \\ \equiv\ & (2\sqrt{3}\sin t + 2\cos t - 2\sqrt{3}\sin t)^2 \\ \equiv\ & 4\cos^2 t \\ \equiv\ & 4 - 4\sin^2 t \\ =\ & 4 - x^2, \text{ as required.} \end{aligned}$$

(b) The x coordinate varies sinusoidally, range $[-2, 2]$. Hence, the lines $x = \pm 2$ are tangent to the curve.

By considering harmonic form, y also varies sinusoidally with amplitude 2, as seen in the Pythagorean sum of $\sqrt{3}$ and 1. So, the lines $y = \pm 2$ are also tangent to the curve.

4424. Throughout, we can ignore the factor $1/\sqrt{2\pi}$, which is a scale factor in the z direction. It doesn't affect behaviour. Let $\varphi(z)$ denote the unscaled version.

(a) Differentiating by the chain rule,

$$\varphi(z) = e^{-\frac{z^2}{2}}$$
$$\implies \varphi'(z) = -ze^{-\frac{z^2}{2}}.$$

This has a factor of z, so $\varphi(z)$ is stationary at $z = 0$.

(b) By the product rule, the second derivative is

$$\varphi''(z) = -e^{-\frac{z^2}{2}} + z^2 e^{-\frac{z^2}{2}}$$
$$\equiv (z^2 - 1)e^{-\frac{z^2}{2}}.$$

This has factors of $(z \mp 1)$, so is zero at $z = \pm 1$.

(c) The second derivative is zero at $z = \pm 1$. Its exponential factor is always positive. Since $(z \mp 1)$ are single factors, $z = \pm 1$ are single roots. Hence, there are sign changes in $\varphi''(z)$ at $z = \pm 1$. So, these are points of inflection.

— Nota Bene —

This also proves, via translations/stretches, the more general result that the probability density function of $X \sim N(\mu, \sigma^2)$ has points of inflection at $\mu \pm \sigma$.

4425. Completing the square, the first circle is

$$(x + 3)^2 + (y - 4)^2 = 25.$$

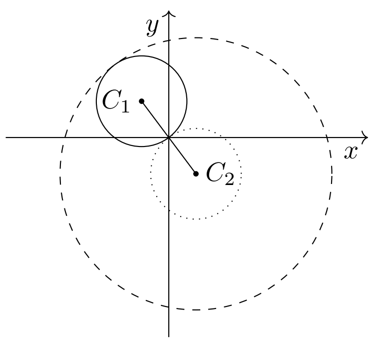
It has centre $(-3, 4)$ and radius 5. The second is

$$(x - 3)^2 + (y + 4)^2 = 25 + \frac{200}{k^2 + 1}.$$

It has centre $(3, -4)$. Its radius depends on

$$k \mapsto 25 + \frac{200}{k^2 + 1}.$$

The range is $(25, 225]$, so the radius can take any value in $(5, 15]$. The boundary cases are as follows, with the circle of radius 5 dotted and the circle of radius 15 dashed. Both are tangent to the first circle:



The second circle is centred on C_2. Its radius is bounded by the circles shown above. Hence, it must intersect the first circle at least once.

4426. There are $6^3 = 216$ outcomes in the possibility space. Of these, the successful triples, classified by the value of X, are

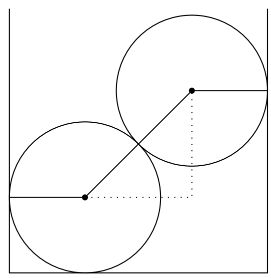
X	1	2	3
	$(1,1,2)$	$(2,2,4)$	$(3,3,6)$
	$(1,2,3)$	$(2,3,5)$	
	$(1,3,4)$	$(2,4,6)$	
	$(1,4,5)$		
	$(1,5,6)$		

Each event of type (A, A, B) has three outcomes; each event of type (A, B, C) has six outcomes. So, there are $3 \times 3 + 6 \times 6 = 45$ successful outcomes. The probability, therefore, is $\frac{45}{216} = \frac{5}{24}$.

4427. Let $x = \tan\theta$. So, $dx = \sec^2\theta\, d\theta$. The new limits are $\theta = 0$ and $\theta = \frac{\pi}{4}$. Enacting the substitution,

$$\begin{aligned}
&\int_0^1 \frac{1}{4x^2 + 4}\, dx \\
&= \int_0^{\frac{\pi}{4}} \frac{1}{4(\tan^2\theta + 1)} \sec^2\theta\, d\theta \\
&= \int_0^{\frac{\pi}{4}} \frac{1}{4\sec^2\theta} \sec^2\theta\, d\theta \\
&= \int_0^{\frac{\pi}{4}} \tfrac{1}{4} d\theta \\
&= \tfrac{\pi}{16}, \text{ as required.}
\end{aligned}$$

4428. (a) Consider the bottom two bottles:



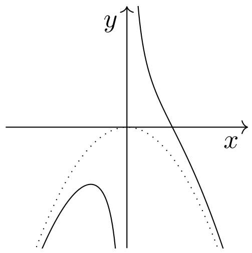
The radius is 5 cm, so the width of the bin is $5 + 5\sqrt{2} + 5 = 10 + 5\sqrt{2}$ cm.

(b) Since all contacts are smooth, the only force supporting bottles $B_{k+1}, ..., B_n$ is the reaction force applied by bottle B_k. The weight of those $n - k$ bottles is $10(n - k)$. So, the vertical component of the reaction must be $10(n - k)$. This gives the magnitude of reaction force as $10\sqrt{2}(n - k)$ N.

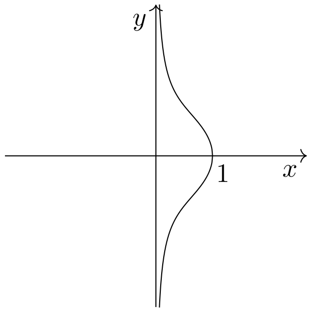
4429. Rearranging, the given equation is $y^2 = 1/x - x^2$. We consider instead the equation $y = \frac{1}{x} - x^2$. We name these as follows:

$$C : y^2 = 1/x - x^2,$$
$$D : y = \frac{1}{x} - x^2.$$

D is the sum of a parabola and reciprocal graph. Its x intercept is at $x = 1$.



C only has points if D has $y \geq 0$. So, there are no points on C outside of $x \in (0, 1]$. Over the domain $x \in (0, 1)$, C is symmetrical in the x axis. So, C is



4430. The first equation is a cubic in $(a + 2b)$:

$$\begin{aligned} &(a+2b)^2\big(1-(a+2b)\big) = 0 \\ \Longrightarrow\ & a + 2b = 0 \text{ or } a + 2b = 1. \end{aligned}$$

Solving each of these with $2a + 5b = 1$ gives (a, b) solution points $(-2, 1)$ and $(3, -1)$.

4431. (a) Setting $y = 0$, the equation is

$$\begin{aligned} &2(10x-1)^2 + 8(5x+2)^2 = 25 \\ \Longrightarrow\ & 400x^2 + 120x + 9 = 0 \\ \Longrightarrow\ & (20x+3)^2 = 0. \end{aligned}$$

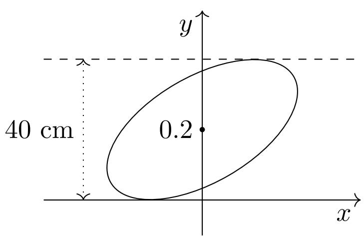
There is a double factor, so the line $y = 0$ is tangent to the curve at $x = -3/20$.

(b) Setting $y = 0.4$,

$$\begin{aligned} &2(10x+1)^2 + 8(5x-2)^2 = 25 \\ \Longrightarrow\ & 400x^2 - 120x + 9 = 0 \\ \Longrightarrow\ & (20x-3)^2 = 0. \end{aligned}$$

This is also a double factor, so the line $y = 0.4$ is tangent to the curve at $x = 3/20$. Since the sculpture is elliptical, this must be a local and global maximum: the sculpture stands 40 cm above the plinth.

(c) Sketching the above facts, the sculpture is



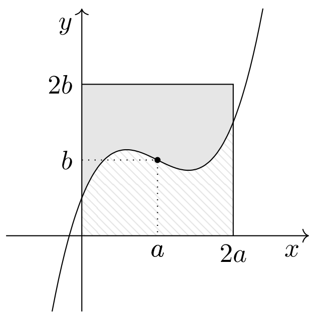
4432. This is true. It makes no difference whether,

① as in $|x^2 - x|$, one final output is rendered positive by a single mod function or,

② as in $|x| \times |x - 1|$, two individual outputs are rendered positive by their own mod functions.

The results are equivalent, so the equations must have the same solution set.

4433. Since the curve $y = \mathrm{f}(x)$ has rotational symmetry around (a, b), the regions shaded below are images of one another.



Therefore, the area of the region between $y = \mathrm{f}(x)$ and the x axis, over the domain $[0, 2a]$, is half the area of the $2a \times 2b$ rectangle shown:

$$\begin{aligned} \int_0^{2a} \mathrm{f}(x)\,dx &= \tfrac{1}{2}(2a \times 2b) \\ &\equiv 2ab, \text{ as required.} \end{aligned}$$

4434. Call the integral I. We integrate by parts twice. Let $u = \sin x$ and $v' = \sin 4x$, so that $u' = \cos x$ and $v = -\frac{1}{4}\cos 4x$. The parts formula gives

$$I = -\tfrac{1}{4}\sin x \cos 4x + \tfrac{1}{4}\int \cos x \cos 4x.$$

And again. Let $u = \cos x$ and $v' = \cos 4x$, so that $u' = -\sin x$ and $v = \frac{1}{4}\sin 4x$. This yields

$$\begin{aligned} I = -\tfrac{1}{4}\sin x \cos 4x &+ \tfrac{1}{16}\cos x \sin 4x \\ &+ \tfrac{1}{16}\int \sin x \sin 4x\,dx. \end{aligned}$$

The last term with the integral is $\frac{1}{16}I$ (up to a constant of integration). So, we can rearrange:

$$\begin{aligned} \tfrac{15}{16}I &= \tfrac{1}{16}\cos x \sin 4x - \tfrac{1}{4}\sin x \cos 4x + c \\ \therefore\quad I &= \tfrac{1}{15}\cos x \sin 4x - \tfrac{4}{15}\sin x \cos 4x + d. \end{aligned}$$

4435. We exponentiate each side of the proposed log law over base a. The LHS gives

$$a^{\log_a x^n} \equiv x^n.$$

Using $(a^p)^q = a^{pq}$, the RHS gives

$$a^{n \log_a x} \equiv (a^{\log_a x})^n \equiv x^n.$$

The LHS and RHS are therefore equal. QED.

4436. (a) $X_1 + X_2 \sim B(2n, p)$.

(b) There are many possible arguments here, any of which will suffice. One argument is: the values modelled by a binomial distribution are non-negative integers. But $X_1 - X_2$ produces negative integers, if e.g. $X_1 = 0$ and $X_2 = 1$. Hence, $X_1 - X_2$ cannot be binomial.

4437. Using $\pm$ compound-angle formulae,

$$\begin{aligned}&\tan\left(\tfrac{\pi}{3} \pm x\right)\\ \equiv\ &\frac{\tan\frac{\pi}{3} \pm \tan x}{1 \mp \tan\frac{\pi}{3}\tan x}\\ \equiv\ &\frac{\sqrt{3} \pm \tan x}{1 \mp \sqrt{3}\tan x}.\end{aligned}$$

So, the RHS of the proposed identity is

$$\begin{aligned}&\tan x \times \frac{\sqrt{3} + \tan x}{1 - \sqrt{3}\tan x} \times \frac{\sqrt{3} - \tan x}{1 + \sqrt{3}\tan x}\\ \equiv\ &\frac{\tan x(3 - \tan^2 x)}{1 - 3\tan^2 x}\\ \equiv\ &\frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}.\end{aligned}$$

The LHS of the proposed identity is

$$\begin{aligned}\tan 3x &\equiv \tan(2x + x)\\ &\equiv \frac{\tan 2x + \tan x}{1 - \tan 2x \tan x}\\ &\equiv \frac{\frac{2\tan x}{1 - \tan^2 x} + \tan x}{1 - \frac{2\tan x}{1 - \tan^2 x}\tan x}\\ &\equiv \frac{2\tan x + \tan x(1 - \tan^2 x)}{1 - \tan^2 x - 2\tan^2 x}\\ &\equiv \frac{3\tan x - \tan^3 x}{1 - 3\tan^2 x}.\end{aligned}$$

The LHS and RHS are equivalent, as required.

4438. The position, velocity and acceleration are

$$\begin{aligned}&\mathrm{g}(t) = \tfrac{1}{2}\mathrm{f}''(a)(t - a)^2 + \mathrm{f}'(a)(t - a) + \mathrm{f}(a)\\ \Longrightarrow\ &\mathrm{g}'(t) = \mathrm{f}''(a)(t - a) + \mathrm{f}'(a)\\ \Longrightarrow\ &\mathrm{g}''(t) = \mathrm{f}''(a).\end{aligned}$$

Substituting $t = a$ into these, each factor of $(t - a)$ is zero, which gives

$$\begin{aligned}\mathrm{g}(a) &= 0 + 0 + \mathrm{f}(a) = \mathrm{f}(a),\\ \mathrm{g}'(a) &= 0 + \mathrm{f}'(a) = \mathrm{f}'(a),\\ \mathrm{g}''(a) &= \mathrm{f}''(a).\end{aligned}$$

Hence, the positions, velocities and accelerations generated by the two models match at $t = a$.

— Nota Bene —

These are the first three terms of the *Taylor series* for $x = \mathrm{f}(t)$ at $t = a$. The generalised binomial expansion and the small-angle approximations are also examples of such series. All are polynomial approximations to non-polynomial functions.

4439. Assuming that θ is a small angle in radians, we can use both a small-angle approximation and the generalised binomial expansion.

$$\begin{aligned}&\left(1 - \sin\theta\right)^{-1}\\ \approx\ &\left(1 - \theta\right)^{-1}\\ =\ &1 + (-1)(-\theta) + \tfrac{(-1)(-2)}{2!}(-\theta)^2 + ...\\ \approx\ &1 + \theta + \theta^2.\end{aligned}$$

4440. (a) The asymptotes are $x = \pm 1$.

(b) Integrating by inspection,

$$\begin{aligned}&\int \frac{x}{\sqrt{1 - x^2}}\,dx\\ =\ &-\tfrac{1}{2}\int \frac{-2x}{\sqrt{1 - x^2}}\,dx\\ =\ &-\sqrt{1 - x^2} + c.\end{aligned}$$

(c) Setting up a limit, the area in question is

$$\begin{aligned}A &= \lim_{p \to 1} \int_0^p \frac{x}{\sqrt{1 - x^2}}\,dx\\ &= \lim_{p \to 1} \left[-\sqrt{1 - x^2}\right]_0^p\\ &= \lim_{p \to 1} \left(-\sqrt{1 - p^2} - (-1)\right)\\ &= \lim_{p \to 1} 1 - \sqrt{1 - p^2}\\ &= 1.\end{aligned}$$

So, despite having infinite extent, the region in question has finite area.

4441. Assume, for a contradiction, that the sequence is a GP. Equating two ratios,

$$\begin{aligned}&\frac{a + b}{b} = \frac{b}{a}\\ \Longrightarrow\ &a^2 + ab = b^2.\end{aligned}$$

Equating another two ratios,

$$\begin{aligned}&\frac{2a + b}{a + b} = \frac{b}{a}\\ \Longrightarrow\ &2a^2 + ab = b^2 + ab\\ \Longrightarrow\ &2a^2 = b^2\\ \Longrightarrow\ &\pm\sqrt{2}a = b.\end{aligned}$$

Substituting this into the first equation,

$$\begin{aligned}&a^2 \pm \sqrt{2}a^2 = 2a^2\\ \Longrightarrow\ &a = 0.\end{aligned}$$

Hence, $a = b = 0$. But this contradicts the fact that the sequence is increasing. So, the sequence cannot be a GP. □

4442. (a) The functions are $f(x) = (x-a)(x-b)$ and $g(x) = (x-b)(x-c)$. So, the given equation is

$$(x-a)(x-b)^2(x-c) = 0.$$

This has three distinct real roots.

(b) Taking out a common factor of $(x-2a)$,

$$\begin{aligned} &(x-a)(x-b) + (x-b)(x-c) = 0 \\ \Longrightarrow\ &(x-b)(2x-a-c) = 0. \end{aligned}$$

This has roots $x_1 = b$ and $x_2 = \frac{a+c}{2}$. These cannot be equal, as x_2 is the arithmetic mean of a and c, which, since $0 < a < c$, is greater than the geometric mean b, according to the AM-GM inequality. Hence, the equation has two distinct real roots.

4443. The variables x and y only appear as x^2 and y^2. Hence, both curves have the x axis and the y axis as lines of symmetry. So, we need only consider the positive quadrant $x, y \geq 0$.

The first equation is $y^2 = 6 - x^2$. Substituting this into the second,

$$\begin{aligned} &x^2(6 - x^2 - x^2) = 4 \\ \Longrightarrow\ &x^4 - 3x^2 + 2 = 0 \\ \Longrightarrow\ &(x^2-2)(x^2-1) = 0. \end{aligned}$$

In the positive quadrant, this gives intersections at $(1, \sqrt{5})$ and $(\sqrt{2}, 2)$. Symmetry dictates that copies of these intersections appear in each of the four quadrants, giving eight points of intersection overall, as required.

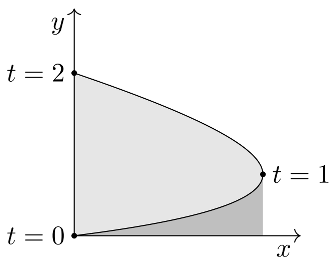
4444. (a) The parametric integration formula is

$$I = \int_{x_1}^{x_2} y\,dx = \int_{t_1}^{t_2} y \frac{dx}{dt}\,dt.$$

Substituting y and $\frac{dx}{dt} = 2 - 2t$,

$$\begin{aligned} I &= \int_0^2 \tfrac{1}{2}(e^{\frac{1}{2}t} - 1)(2-2t)\,dt \\ &= \int_0^2 (e^{\frac{1}{2}t} - 1)(1-t)\,dt, \text{ as required.} \end{aligned}$$

(b) At $t = 0$, the coordinates are $(0, 0)$; at $t = 2$, they are $(0, e/2 - 1)$. The point at which the gradient is parallel to y is $t = 1$. For $t \in [0, 1)$, $\frac{dx}{dt} > 0$, so the integral calculates positive area; for $t \in (1, 2]$, $\frac{dx}{dt} < 0$, so the integral calculates negative area.



But, as can be seen, the negative area (total shaded) is greater than the positive (darker shaded), since y is increasing for all $t \in [0, 2]$.

(c) We integrate by parts. Let $u = 1 - t$ and $v' = e^{\frac{1}{2}t} - 1$, so that $u' = -1$ and $v = 2e^{\frac{1}{2}t} - t$. The parts formula gives

$$\begin{aligned} I &= \Big[(2e^{\frac{1}{2}t} - t)(1-t)\Big]_0^2 + \int_0^2 2e^{\frac{1}{2}t} - t\,dt \\ &= \Big[(2e^{\frac{1}{2}t} - t)(1-t) + 4e^{\frac{1}{2}t} - \tfrac{1}{2}t^2\Big]_0^2 \\ &= 2e - 6. \end{aligned}$$

This is negative, as expected. So, the area of the shaded region is $6 - 2e$.

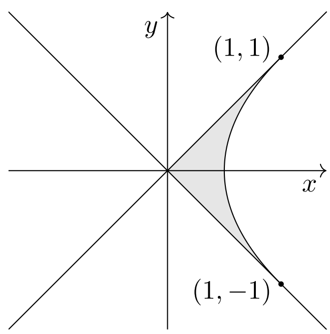
4445. Working with a tetrahedron of side length 1, the base has perpendicular height $\sqrt{3}/2$, so the centre of the base is $\sqrt{3}/3$ away from its vertices. Forming a vertical right-angled triangle, the height is $\sqrt{6}/3$. Scaling up by l and equating to 2,

$$\begin{aligned} &\tfrac{\sqrt{6}}{3} l = 2 \\ \Longrightarrow\ &l = \sqrt{6}. \end{aligned}$$

4446. Squaring the equation of the curve,

$$\begin{aligned} &\sqrt{x+y} + \sqrt{x-y} = 1 \\ \Longrightarrow\ &x + y + 2\sqrt{x^2 - y^2} + x - y = 1 \\ \Longrightarrow\ &2\sqrt{x^2-y^2} = 1 - 2x \\ \Longrightarrow\ &4x^2 - 4y^2 = 1 - 4x + 4x^2 \\ \Longrightarrow\ &x = \tfrac{1}{4} + y^2. \end{aligned}$$

The region in question is



Integrating with respect to y,

$$\begin{aligned} A &= \int_{-\frac{1}{2}}^{\frac{1}{2}} \tfrac{1}{4} + y^2\,dy - \tfrac{1}{4} \\ &= \Big[\tfrac{1}{4}y + \tfrac{1}{3}y^3\Big]_{-\frac{1}{2}}^{\frac{1}{2}} - \tfrac{1}{4} \\ &= 2\big(\tfrac{1}{8} + \tfrac{1}{24}\big) - \tfrac{1}{4} \\ &= \tfrac{1}{12}, \text{ as required.} \end{aligned}$$

4447. The factorisation is

$$3x^3 - 2x^2y^2 + 3xy - 2y^3 \equiv (x^2 + y)(3x - 2y^2).$$

Written as a division, this is

$$\frac{3x^3 - 2x^2y^2 + 3xy - 2y^3}{x^2 + y} \equiv 3x - 2y^2.$$

4448. Putting everything on the LHS,

$$\begin{aligned} & \mathrm{f}(x)^2 + \mathrm{g}(x)^2 = 2\,\mathrm{f}(x)\,\mathrm{g}(x) \\ \iff & \mathrm{f}(x)^2 - 2\,\mathrm{f}(x)\,\mathrm{g}(x) + \mathrm{g}(x)^2 = 0 \\ \iff & \big(\mathrm{f}(x) - \mathrm{g}(x)\big)^2 = 0 \\ \iff & \mathrm{f}(x) - \mathrm{g}(x) = 0 \\ \iff & \mathrm{f}(x) = \mathrm{g}(x). \end{aligned}$$

Hence, A is the solution set of both equations.

— Nota Bene —

We need implication in both directions $\iff$ to show that the solution sets are the same. If, in the algebra above, we used only forwards implication $\implies$, then we would only have proved that A is a *subset* of the solution set of the second equation.

4449. The derivative is $1 - 1/x^2$. So, at a generic point $(p, p + 1/p)$, the gradient is $1 - 1/p^2$. Hence, the equation of the tangent is

$$y - p - \tfrac{1}{p} = \left(1 - \tfrac{1}{p^2}\right)(x - p).$$

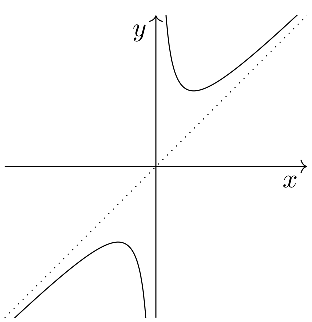
Substituting $y = x + \frac{1}{x}$,

$$\begin{aligned} & x + \tfrac{1}{x} - p - \tfrac{1}{p} = \left(1 - \tfrac{1}{p^2}\right)(x - p) \\ \implies & x + \tfrac{1}{x} - p - \tfrac{1}{p} = x - p - \tfrac{x}{p^2} + \tfrac{1}{p} \\ \implies & \tfrac{x}{p^2} - \tfrac{2}{p} + \tfrac{1}{x} = 0 \\ \implies & x^2 - 2xp + p^2 = 0 \\ \implies & (x - p)^2 = 0. \end{aligned}$$

So, there is a point of tangency at $x = p$, which we already knew, and no others. Therefore, the tangent does not re-intersect the curve.

— Alternative Method —

Graphically, this fact can be seen from a sketch of the curve $y = x + \frac{1}{x}$. The graph has a vertical asymptote at $x = 0$ and an oblique asymptote at $y = x$.



Without loss of generality, assume the tangent is drawn to a point in the positive quadrant.

- The tangent cannot re-intersect the curve in the positive quadrant, since the curvature is positive for $x \in (0, \infty)$,
- The tangent cannot re-intersect the curve in the negative quadrant, since no such tangent passes into the region defined by $y < x < 0$.

So, no tangent re-intersects the curve.

4450. For k variables to change, the number available for change must be in $\{k, k+1, ..., n\}$. We express this as integer values of r from $r = k$ to $r = n$.

The probability that r variables are available for change is $\frac{1}{n+1}$, as there are $n + 1$ elements of the set $\{0, 1, ..., n\}$.

Once r is chosen, the distribution of the number of variables that do change is $Y \sim B(r, 1/2)$. So, the probability that exactly k variables change, given r available to change, is

$$\mathbb{P}(Y = k) = {}^r\mathrm{C}_k \tfrac{1}{2}^{r-k} \tfrac{1}{2}^k = \frac{{}^r\mathrm{C}_k}{2^r}.$$

So, the probability that r variables are available and exactly k of them change is

$$\frac{1}{n+1} \times \frac{{}^r\mathrm{C}_k}{2^r}.$$

Adding this up over all values of r, starting with $r = k$ and finishing with $r = n$,

$$\begin{aligned} p &= \sum_{r=k}^{n} \frac{1}{n+1} \times \frac{{}^r\mathrm{C}_k}{2^r} \\ &\equiv \frac{1}{n+1} \sum_{r=k}^{n} \frac{{}^r\mathrm{C}_k}{2^r}, \text{ as required.} \end{aligned}$$

4451. Heron's formula gives

$$A^2 = s(s - a)^2(s - b).$$

We know that $2s = 2a + b$, so $b = 2s - 2a$. Subbing this in,

$$\begin{aligned} A^2 &= s(s - a)^2\big(s - (2s - 2a)\big) \\ &\equiv (s - a)^2(2as - s^2). \end{aligned}$$

Differentiating with respect to a,

$$\begin{aligned} \frac{d}{da}(A^2) &= -2(s - a)(2as - s^2) + 2s(s - a)^2 \\ &\equiv 2s(s - a)(2s - 3a). \end{aligned}$$

Setting this to zero, the area is stationary at $a = s$ and $a = \frac{2}{3}s$. We reject $a = s$, as it gives $b = 0$. So, the area is maximised at $a = \frac{2}{3}s$, which gives $b = \frac{2}{3}s$. This is an equilateral triangle. □

4452. This is almost correct. However, the value $k = 0$ is a counterexample. $\bar{X}$ is continuous, so

$$\mathbb{P}\left(|\bar{X} - \mu| > 0\right) = 1.$$

Letting n tend to infinity doesn't change the value 1, which means that $k = 0$ disproves the result. However, the result is true for any constant $k \neq 0$.

4453. The angle of rotation is 45°, so the dashed line in the diagram is angled at 22.5° away from the y axis. Its equation is therefore

$$\begin{aligned} y &= x \tan 67.5° \\ &\equiv \big(1 + \sqrt{2}\big)x. \end{aligned}$$

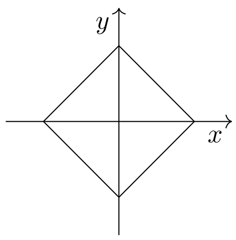
Solving this simultaneously with $y = x^2$, the point of intersection is

$$\big(1 + \sqrt{2}, 3 + 2\sqrt{2}\big).$$

So, the area in question is

$$\begin{aligned} A &= 2\int_0^{1+\sqrt{2}} \big(1 + \sqrt{2}\big)x - x^2\,dx \\ &= 2\Big[\tfrac{1}{2}\big(1 + \sqrt{2}\big)x^2 - \tfrac{1}{3}x^3\Big]_0^{1+\sqrt{2}} \\ &= \tfrac{1}{3}\big(1 + \sqrt{2}\big)^3 \\ &= \tfrac{1}{3}\big(7 + 5\sqrt{2}\big), \text{ as required.} \end{aligned}$$

4454. In the positive quadrant, the locus is $x + y = 1$, which is a straight line segment between $(0, 1)$ and $(1, 0)$. The same appears symmetrically in each quadrant. So, the locus is a square:



Its side length is $\sqrt{2}$, so its area is 2 square units.

4455. The first four terms of the proposed Cartesian equation may be factorised, giving the LHS as

$$(x + y)^3 + 18x - 9y.$$

Substituting the parametric equations, this is

$$\begin{aligned} &\big(t - t^3 + 2t + t^3\big)^3 + 18\big(t - t^3\big) - 9\big(2t + t^3\big) \\ &\equiv (3t)^3 + 18t - 18t^3 - 18t - 9t^3 \\ &\equiv 27t^3 - 18t^3 - 9t^3 \\ &\equiv 0, \text{ as required.} \end{aligned}$$

4456. Differentiating the trial solution,

$$\begin{aligned} \frac{dy}{dx} &= kx^{k-1} \\ \implies \frac{d^2y}{dx^2} &= k(k-1)x^{k-2}. \end{aligned}$$

Substituting these into the DE,

$$\begin{aligned} &x^2k(k-1)x^{k-2} + axkx^{k-1} + bx^k = 0 \\ \implies &k(k-1)x^k + akx^k + bx^k = 0 \\ \implies &\big(k(k-1) + ak + b\big)x^k = 0. \end{aligned}$$

This needs to hold for all x. So,

$$\begin{aligned} &k(k-1) + ak + b = 0 \\ \implies &k^2 + (a-1)k + b = 0, \text{ as required.} \end{aligned}$$

4457. Writing in partial fractions,

$$\frac{x^3 - 7x^2 + 14x - 7}{x^2 - 7x + 12} \equiv x + \frac{1}{x-3} + \frac{1}{x-4}.$$

Next, we expand binomially. Since x is small, we can neglect terms in x^2 or higher:

$$\begin{aligned} \frac{1}{x-3} &\approx -\tfrac{1}{3} - \tfrac{1}{9}x, \\ \frac{1}{x-4} &\approx -\tfrac{1}{4} - \tfrac{1}{16}x. \end{aligned}$$

This gives

$$\begin{aligned} &\frac{x^3 - 7x^2 + 14x - 7}{x^2 - 7x + 12} \\ &\approx x - \tfrac{1}{3} - \tfrac{1}{9}x - \tfrac{1}{4} - \tfrac{1}{16}x \\ &\equiv \frac{119x - 84}{144}, \text{ as required.} \end{aligned}$$

4458. Let $u = 1 + e^{2x}$, so that $du = 2e^{2x}\,dx$. This appears directly if we factorise the numerator:

$$\begin{aligned} &\int \frac{2e^{4x}}{1 + e^{2x}}\,dx \\ &= \int \frac{e^{2x}}{1 + e^{2x}} 2e^{2x}\,dx \end{aligned}$$

Enacting the substitution, this is

$$\begin{aligned} &\int \frac{u - 1}{u}\,du \\ &= \int 1 - \frac{1}{u}\,du \\ &= u - \ln|u| + c \\ &= e^{2x} - \ln\big|1 + e^{2x}\big| + c. \end{aligned}$$

4459. Reflection in the line $x = k$ is equivalent to

- reflection in $x = 0$ to give $y = \mathrm{f}(-x)$, then
- translation by $2k\mathbf{i}$, to give $y = \mathrm{f}\big(-(x - 2k)\big)$, which simplifies to $y = \mathrm{f}(2k - x)$.

Rotation 90° anticlockwise around O is

- reflection in $x = 0$ to give $y = \mathrm{f}\big(2k - (-x)\big)$, which simplifies to $y = \mathrm{f}(2k + x)$, then
- reflection in $y = x$, to give $x = \mathrm{f}(2k + y)$.

4460. The average speed over the first k seconds is

$$\begin{aligned}\bar{v} &= \tfrac{1}{k}\int_0^k at^2 + b\,dt \\ &\equiv \tfrac{1}{k}\left[\tfrac{1}{3}at^3 + bt\right]_0^k \\ &\equiv \tfrac{1}{3}ak^3 + b.\end{aligned}$$

Setting this equal to the instantaneous speed,

$$\begin{aligned}& aT_0^2 + b = \tfrac{1}{3}ak^3 + b \\ \Longrightarrow\ & T_0^2 = \tfrac{1}{3}k^3.\end{aligned}$$

T_0 is independent of a and b, as required.

4461. Consider the indefinite integral

$$I = \int \sqrt{x}\ln x\,dx.$$

We integrate by parts. Let $u = \ln x$ and $v' = \sqrt{x}$, so that $u' = \frac{1}{x}$ and $v = \frac{2}{3}x^{\frac{3}{2}}$. This gives (leaving aside the constant of integration)

$$\begin{aligned}I &= \tfrac{2}{3}x^{\frac{3}{2}}\ln x - \int \tfrac{2}{3}x^{\frac{1}{2}}\,dx \\ &= \tfrac{2}{3}x^{\frac{3}{2}}\ln x - \tfrac{4}{9}x^{\frac{3}{2}}.\end{aligned}$$

We can't evaluate I at the lower limit $x = 0$, because $\ln x$ is undefined. However, as $x \to 0$, $x^{\frac{3}{2}}\ln x \to 0$, as can be verified by checking small values. So, the lower limit produces 0, and the equation in question is therefore

$$\begin{aligned}& \tfrac{2}{3}k^{\frac{3}{2}}\ln k - \tfrac{4}{9}k^{\frac{3}{2}} = 0 \\ \Longrightarrow\ & k^{\frac{3}{2}}\left(\ln k - \tfrac{2}{3}\right) = 0 \\ \Longrightarrow\ & k = 0, e^{\frac{2}{3}}.\end{aligned}$$

4462. Rewriting the sum,

$$\begin{aligned}\sum_{r=1}^{n}(2r-1)^2 &\equiv \sum_{r=1}^{2n} r^2 - \sum_{r=1}^{n}(2r)^2 \\ &\equiv \sum_{r=1}^{2n} r^2 - 4\sum_{r=1}^{n} r^2.\end{aligned}$$

Using the given result, this is

$$\begin{aligned}& \tfrac{1}{6}(2n)(2n+1)(4n+1) - 4\big(\tfrac{1}{6}n(n+1)(2n+1)\big) \\ \equiv\ & \tfrac{1}{3}n(2n+1)\big((4n+1) - 2(n+1)\big) \\ \equiv\ & \tfrac{1}{3}n(2n+1)(2n-1), \text{ as required.}\end{aligned}$$

4463. Assume, for a contradiction, that $x = p/q$, where $p, q \in \mathbb{Z}$, and $2^{x+1} - 3^{x-1} = 0$. This gives

$$\begin{aligned}& 2^{\frac{p}{q}+1} = 3^{\frac{p}{q}-1} \\ \Longrightarrow\ & 2^{\frac{p+q}{q}} = 3^{\frac{p-q}{q}} \\ \Longrightarrow\ & 2^{p+q} = 3^{p-q}.\end{aligned}$$

Since 2 and 3 are coprime and $p, q \in \mathbb{Z}$, this can only hold if both sides are equal to 1. Hence $p + q = 0$ and $p - q = 0$, which gives $p = q = 0$. The contradicts the fact that $x = p/q$ is a root of the original equation. Hence, if x is a root of $2^{x+1} - 3^{x-1} = 0$, then $x \notin \mathbb{Q}$. □

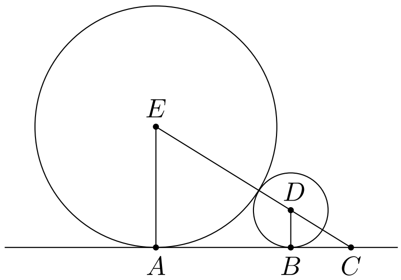
4464. (a) The polynomial $\mathrm{f}(x) - \mathrm{g}(x)$ has degree at most $2k$. This is even, so $n \in \{0, 1, ..., 2k\}$.

(b) The polynomial $\mathrm{f}(x) - \mathrm{g}(x)$ has degree at most $2k + 1$. The boundary case is odd, suggesting $n \in \{1, ..., 2k + 1\}$. But it is not guaranteed that $\mathrm{f}(x) - \mathrm{g}(x)$ has degree $2k+1$. For example $\mathrm{f}(x) = x^3 + 1$ and $\mathrm{g}(x) = x^3 + 2$. So, zero is also possible: $n \in \{0, 1, ..., 2k + 1\}$.

(c) The polynomial $\mathrm{f}(x) - \mathrm{g}(x)$ has degree $2k$, with certainty. This is even, so $n \in \{0, 1, ..., 2k\}$.

(d) Again with certainty, $\mathrm{f}(x) - \mathrm{g}(x)$ has degree $2k + 1$. This is odd, so $n \in \{1, ..., 2k + 1\}$.

4465. Labelling various points, the scenario is



Triangles AEC and BDC are similar, in the ratio $81 : 25$. Also $|ED| = \frac{1}{25} + \frac{1}{81} = \frac{106}{2025}$. So,

$$\begin{aligned}& \tfrac{25}{81}|CE| = |CE| - \tfrac{106}{2025} \\ \Longrightarrow\ & |CE| = \tfrac{53}{700}.\end{aligned}$$

This gives $|CD| = \frac{53}{700} - \frac{106}{2025} = \frac{53}{2268}$, and also

$$\begin{aligned}|AC| &= \sqrt{|CE|^2 - |AE|^2} \\ &= \tfrac{9}{140}.\end{aligned}$$

Putting these facts together,

$$\begin{aligned}|AB| &= |ED| \times \frac{|AC|}{|CE|} \\ &= \tfrac{106}{2025} \times \frac{\frac{9}{140}}{\frac{53}{700}} \\ &= \tfrac{2}{45}, \text{ as required.}\end{aligned}$$

4466. The individual periods are

$\sin x$	2π	$\sin 4x$	$\frac{\pi}{2}$
$\cos x$	2π	$\cos 5x$	$\frac{2\pi}{5}$
$\tan x$	π	$\tan 6x$	$\frac{\pi}{6}$

The period of f is the lowest common multiple of the individual periods. Since 2 and 5 are coprime, this is 2π.

4467. Differentiating implicitly, using the product rule,

$$\begin{aligned}
&\frac{d}{dx}(xy) = y + x\frac{dy}{dx}\\
\Longrightarrow\ &\frac{d^2}{dx^2}(xy) = \frac{dy}{dx} + \frac{dy}{dx} + x\frac{d^2y}{dx^2}\\
&\qquad\equiv 2\frac{dy}{dx} + x\frac{d^2y}{dx^2}\\
\Longrightarrow\ &\frac{d^3}{dx^3}(xy) = 2\frac{d^2y}{dx^2} + \frac{d^2y}{dx^2} + x\frac{d^3y}{dx^2}\\
&\qquad\equiv 3\frac{d^2y}{dx^2} + x\frac{d^3y}{dx^2}, \text{ as required.}
\end{aligned}$$

4468. Renaming one of the indexing variables, a number appears in both sequences iff

$$3 \times 2^{m-1} = 4 \times 3^{n-2}.$$

Consider the prime factorisations of the LHS and RHS. The LHS has exactly one factor of three, so the RHS must do too, giving $n = 2$. The RHS has exactly two factors of two, so the LHS must do too, giving $m = 3$. Therefore, the number 12 appears in both sequences, but no other numbers do.

4469. (a) As in the identity $\cos x \equiv \sin\left(\frac{\pi}{2} - x\right)$, the sin and cos functions are symmetrical around the input $\frac{\pi}{4}$. So, $x = \frac{\pi}{4}$ must be either a minimum or maximum. Since $\sin x + \cos x$ is maximised (rather than minimised) at this value, $f(x)$ has a local maximum at $x = \frac{\pi}{4}$.

(b) Evaluating the function,

$$f\left(\tfrac{\pi}{4}\right) = 4.056... > 4.$$

At its minimum,

$$f\left(\tfrac{5\pi}{4}\right) = 1 < 4.$$

So, there must be (at least) two roots of $f(x) - 4 = 0$ for $x \in [0, 2\pi)$. And, since $\sin x$ and $\cos x$ are both periodic with period 2π, copies of these same two roots must appear in each domain of the form

$$[2n\pi, 2(n+1)\pi).$$

Hence, $f(x) - 4 = 0$ has infinitely many roots, as required.

4470. (a) Subbing the parametrics into the LHS,

$$\begin{aligned}
&x^{\frac{2}{3}} + y^{\frac{2}{3}}\\
&= (a\cos^3 t)^{\frac{2}{3}} + (a\sin^3 t)^{\frac{2}{3}}\\
&\equiv a^{\frac{2}{3}}(\cos^2 t + \sin^2 t)\\
&\equiv a^{\frac{2}{3}}, \text{ as required.}
\end{aligned}$$

(b) Differentiating,

$$\frac{dx}{dt} = -3\cos^2 t \sin t.$$

So, in the positive quadrant, the area is

$$\begin{aligned}
&\int_{t_1}^{t_2} y\frac{dx}{dt}\,dt\\
&= \int_{\frac{\pi}{2}}^{0} a\sin^3 t \cdot -3\cos^2 t \sin t\,dt\\
&= 3a\int_0^{\frac{\pi}{2}} \sin^4 t\cos^2 t\,dt.
\end{aligned}$$

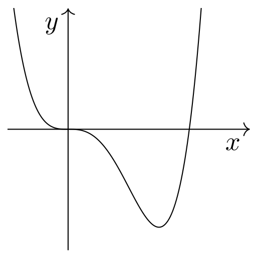
Using the integration facility on a calculator, this is $0.098175...a^2$. Taking out a factor of π gives $\frac{3}{32}\pi a^2$. So, the total area enclosed by the astroid is $\frac{3}{8}\pi a^2$.

4471. (a) Differentiating with $\varepsilon = 0$,

$$\begin{aligned}
&y = 3x^4 - 12x^3 = 3x^3(x-4)\\
\Longrightarrow\ &\frac{dy}{dx} = 12x^3 - 36x^2 \equiv 12x^2(x-3)\\
\Longrightarrow\ &\frac{d^2y}{dx^2} \equiv 36x^2 - 72x = 36x(x-2).
\end{aligned}$$

The first derivative is zero at the origin, so the origin is a stationary point. The second derivative has a single root at $x = 0$, so it is zero and changes sign. Therefore, the origin is a stationary point of inflection.

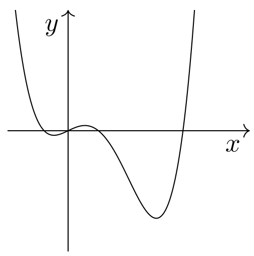
(b) The curve is a positive quartic with a triple root at $x = 0$ and a single root at $x = 4$. It has another stationary point at $x = 3$.



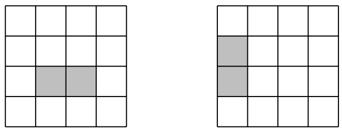
(c) Let $\varepsilon > 0$. For small x, the quartic term can be neglected in favour of the cubic and linear terms. This gives, in the vicinity of the origin, $y = -12x^3 + \varepsilon x$. Setting this to zero,

$$\begin{aligned}
&-12x^3 + \varepsilon x = 0\\
\Longrightarrow\ &x = 0, \pm\sqrt{\tfrac{\varepsilon}{12}}.
\end{aligned}$$

This gives three roots in the vicinity of and approximately symmetrical around the origin. The root around $x = 4$ remains, since, in that region, εx is negligible compared to the quartic and cubic terms.



4472. There are 16! outcomes. For successful outcomes, 1 and 2 form a rectangle.



Horizontally, there are $4 \times 3 = 12$ locations for the rectangle. There are also 12 vertically, giving 24 overall. For each of these, 1 and 2 can be placed in 2! ways, and the remaining 14 numbers in 14! ways. So,

$$p = \frac{24 \cdot 2! \cdot 14!}{16!} = \tfrac{1}{5}.$$

— Alternative Method —

Choosing a spot for 1, there are three cases:

① With probability $\frac{1}{4}$, 1 is in the middle. The probability that 2 is placed alongside is $\frac{4}{15}$.

② With probability $\frac{1}{2}$, 1 is on an edge, but not in a corner. The probability that 2 is placed alongside is $\frac{3}{15}$.

③ With probability $\frac{1}{4}$, 1 is in a corner. The probability that 2 is placed alongside is $\frac{2}{15}$.

So, the probability is

$$p = \tfrac{1}{4} \times \tfrac{4}{15} + \tfrac{1}{2} \times \tfrac{3}{15} + \tfrac{1}{4} \times \tfrac{2}{15} = \tfrac{1}{5}.$$

4473. We need to integrate by parts twice, so we use the tabular integration method. The relevant table is

Signs	Derivatives	Integrals
$+$	x^2	$\sin x$
$-$	$2x$	$-\cos x$
$+$	2	$-\sin x$
$-$	0	$\cos x$.

The indefinite integral is

$$\begin{aligned}&\int x^2 \sin x \, dx \\ &= -x^2 \cos x + 2x \sin x + 2\cos x + c \\ &\equiv \left(2 - x^2\right)\cos x + 2x \sin x + c.\end{aligned}$$

So, the definite integral is

$$\begin{aligned}&\int_0^{\frac{\pi}{2}} x^2 \sin x \, dx \\ &= \Big[\left(2 - x^2\right)\cos x + 2x \sin x\Big]_0^{\frac{\pi}{2}} \\ &= \pi - 2, \text{ as required.}\end{aligned}$$

4474. (a) Applying the function twice,

$$\mathrm{f}^2 : x \longmapsto k - x \longmapsto k - (k - x).$$

Since $k-(k-x) \equiv x$, we know that $\mathrm{f}^2 : x \mapsto x$. Hence, f is self-inverse, as required.

(b) If f is a polynomial of degree n, then f^2 (the composition of f with itself) has degree n^2. For a function to be self-inverse, it is necessary that $\mathrm{f}^2 : x \longmapsto x$. This has degree 1. So, f can only be self-inverse if $n^2 = 1$. Hence, no polynomial function of degree 2 or greater is self-inverse.

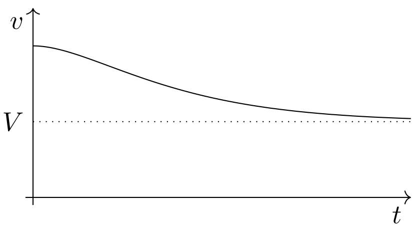
4475. (a) The factor e^{-kt} tends to zero, and dominates $(t + \frac{1}{k})$. Hence, $v \to V$. So, the constant V is terminal velocity.

(b) Differentiating with respect to t,

$$\begin{aligned}a &= -Ake^{-kt}\left(t + \tfrac{1}{k}\right) + Ae^{-kt} \\ &\equiv -Akte^{-kt}.\end{aligned}$$

At $t = 0$, the predicted acceleration is $a = 0$.

(c) On the velocity-time graph, the initial gradient is zero. The velocity then decreases, tending asymptotically to V.



(d) Differentiating the acceleration,

$$\frac{da}{dt} = -Ake^{-kt} + Ak^2te^{-kt}.$$

Setting this to zero,

$$\begin{aligned}&Ae^{-kt}(-k + k^2t) = 0 \\ \Longrightarrow\ &t = \tfrac{1}{k}.\end{aligned}$$

So, the maximum deceleration occurs at $t = \frac{1}{k}$, and has magnitude $|a| = Ae^{-1}$.

4476. Put into harmonic form, the RHS is $2\sin\left(x - \frac{\pi}{6}\right)$. Over the domain $[0, 2\pi]$, the relevant values (roots, maximum and minimum) are as follows:

x	$\frac{\pi}{6}$	$\frac{2\pi}{3}$	$\frac{7\pi}{6}$	$\frac{5\pi}{3}$
$\sqrt{y}$	0	2	0	-2.

The negative value -2 cannot equal the output of $\sqrt{y}$. So, the last of these does not produce a point on the original graph. The first three correspond to the SPs required. Squaring $\sqrt{y}$, the SPs of the original graph are

x	$\frac{\pi}{6}$	$\frac{2\pi}{3}$	$\frac{7\pi}{6}$
y	0	4	0
Type	Min	Max	Min

— Alternative Method —

Put into harmonic form, the RHS is $2\sin\left(x-\frac{\pi}{6}\right)$. Squaring the equation (noting the introduction of new points),

$$\begin{aligned} y &= 4\sin^2\left(x-\tfrac{\pi}{6}\right) \\ \implies \frac{dy}{dx} &= 8\sin\left(x-\tfrac{\pi}{6}\right)\cos\left(x-\tfrac{\pi}{6}\right) \\ &\equiv 4\sin\left(2x-\tfrac{\pi}{3}\right). \end{aligned}$$

Setting the first derivative to zero for SPs,

$$\begin{aligned} &4\sin(2x-\tfrac{\pi}{3}) = 0 \\ \implies &2x-\tfrac{\pi}{3} = 0, \pi, 2\pi, 3\pi, ... \\ \implies &2x = \tfrac{\pi}{3}, \tfrac{4\pi}{3}, \tfrac{7\pi}{3}, \tfrac{10\pi}{3}, ... \\ \implies &x = \tfrac{\pi}{6}, \tfrac{2\pi}{3}, \tfrac{7\pi}{6}, \tfrac{5\pi}{3}, ... \end{aligned}$$

The first three of these x values produce values of y. The last doesn't. This gives three SPs:

x	$\frac{\pi}{6}$	$\frac{2\pi}{3}$	$\frac{7\pi}{6}$
y	0	4	0
Type	Min	Max	Min.

4477. The implication goes backwards. If $\text{f}(x)-\text{g}(x)$ has a factor of $(x-a)^2$, then $\text{f}(x)-\text{g}(x)$ has a double root at $x=a$. Hence, $\text{f}(x)-\text{g}(x)$ is stationary at $x=a$, meaning that $\text{f}'(a)=\text{g}'(a)$.

But the converse is not true. A counterexample is $\text{f}(x)=x^2$ and $\text{g}(x)=x-1$, with $a=0$. We have $\text{f}'(0)=\text{g}'(0)=0$, but $\text{f}(x)-\text{g}(x)=x^2-x+1$, which does not have a factor of x^2.

4478. (a) Testing the input $-x$,

$$\begin{aligned} \text{A}(-x) &= \text{f}(-x)+\text{f}(x) \\ &\equiv \text{f}(x)+\text{f}(-x) \\ &= \text{A}(x). \end{aligned}$$

So, A is even. Also

$$\begin{aligned} \text{B}(-x) &= \text{f}(-x)-\text{f}(x) \\ &\equiv -(\text{f}(x)-\text{f}(-x)) \\ &= -\text{B}(x). \end{aligned}$$

So, B is odd.

(b) Taking the mean of the functions A and B,

$$\tfrac{1}{2}\text{A}(x)+\tfrac{1}{2}\text{B}(x)=\text{f}(x).$$

This is a decomposition of $\text{f}(x)$ into an even and an odd function, which proves the result by construction.

4479. By definition, $\sec\theta=\frac{1}{\cos\theta}$. Using a small-angle approximation for $\cos\theta$,

$$\sec\theta \approx \left(1-\tfrac{1}{2}\theta^2\right)^{-1}.$$

We use the generalised binomial expansion. Since θ is small, we can neglect terms in θ^4 and above. This gives $\sec\theta$ as approximately equal to

$$\begin{aligned} &1+(-1)\left(-\tfrac{1}{2}\theta^2\right) \\ \equiv\ &1+\tfrac{1}{2}\theta^2 \\ \equiv\ &2-\left(1-\tfrac{1}{2}\theta^2\right) \\ \approx\ &2-\cos\theta \\ =\ &2-\sqrt{1-\sin^2\theta} \\ \approx\ &2-\sqrt{1-\theta^2}, \text{ as required.} \end{aligned}$$

— Alternative Method —

Expanding the RHS binomially, again neglecting terms in θ^4 and above,

$$\begin{aligned} &2-\left(1-\theta^2\right)^{\frac{1}{2}} \\ =\ &2-\left(1+\tfrac{1}{2}\left(-\theta^2\right)+...\right) \\ \equiv\ &2-\left(1-\tfrac{1}{2}\theta^2+...\right) \\ \equiv\ &1+\tfrac{1}{2}\theta^2+... \end{aligned}$$

This agrees with the expansion of $\sec\theta$ given in the first solution.

4480. (a) The second equation is $y=x^{\frac{2}{3}}$. This gives

$$\begin{aligned} &xe^x = x^{\frac{2}{3}} \\ \implies &x=0 \text{ or } e^x = x^{-\frac{1}{3}}. \end{aligned}$$

Raising the latter to the power -3, $e^{-3x}=x$. Since α is non-zero, this gives $\alpha=e^{-3\alpha}$. So, α is a fixed point of $x_{n+1}=e^{-3x_n}$.

(b) The N-R iteration for $x-e^{-3x}=0$ is

$$x_{n+1}=x_n-\frac{x_n-e^{-3x_n}}{1+3e^{-3x_n}}.$$

Running this with $x_0=0.5$, $x_1=0.3341...$, then $x_n\to 0.3499....$ So, $\alpha=0.350$ (3sf).

(c) In a fixed-point iteration $x_{n+1}=\text{g}(x_n)$, the condition for convergence is $|\text{g}'(\alpha)|<1$. For the iteration in (a), $\text{g}'(x)=-3e^{-3x}$. At the fixed point, $\text{g}'(\alpha)=-1.0499....$ This is less than -1, so the iteration diverges. However, since it is only just less than -1, the iteration diverges very slowly.

— Nota Bene —

Consider the inverse iteration

$$x_{n+1}=\ln\left(x^{-\frac{1}{3}}\right).$$

Its gradient $\text{g}'(\alpha)$ is the reciprocal of that of the iteration in (a), i.e. $1/-1.05\approx -0.95$. So, while one diverges, but very slowly, the other converges, but very slowly. This is one of the reasons why the N-R iteration is so much more reliable than fixed-point iteration.

4481. For x intercepts, $x(e^{2x}+k)=0$, so $x=0$ or $x=\frac{1}{2}\ln(-k)$. For SPs,

$$(e^{2x}+k)+2xe^{2x}=0.$$

Substituting $x=0$ gives $1+k=0$, so $k=-1$. Alternatively, substituting $x=\frac{1}{2}\ln(-k)$,

$$\begin{aligned}-k\ln(-k)&=0\\ \Longrightarrow k&=0,-1.\end{aligned}$$

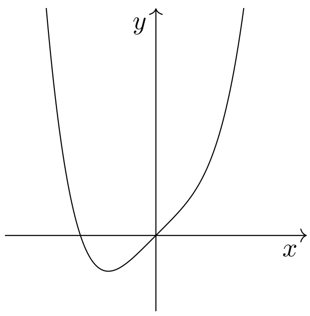
The value $k=0$ does not give a stationary point on the x axis, as $\ln 0$ is undefined. So, $k=-1$.

4482. (a) The solution set of $\mathrm{g}(x)\,\mathrm{h}(x)\geq 0$ is $P\cap Q$, which does not contain $P'\cap Q$.

(b) The solution set is $(P\cap Q')\cup(P'\cap Q)$, which does contain $P'\cap Q$.

(c) The exact solution set of $\mathrm{g}(x)+\mathrm{h}(x)>0$ can't be determined from the information given. However, for any element $x\in P\cap Q$, $\mathrm{g}(x)\geq 0$ and $\mathrm{h}(x)\geq 0$, so $\mathrm{g}(x)+\mathrm{h}(x)\geq 0$. Therefore, the solution set does include $P\cap Q$.

4483. A counterexample is $\mathrm{f}(x)=x^4+x$, with $p=1$ and $q=-1$. The second derivative is $\mathrm{f}''(x)=12x^2$. So, $\mathrm{f}''(1)=12$ and $\mathrm{f}''(-1)=12$. But the curve doesn't have the y axis as a line of symmetry:



4484. Equation A factorises as $(z-1)(x+y)=0$, which is satisfied if $z=1$ or $x+y=0$.

Equation B holds if $z=1$ or $x+y=1$.

If we suppose that $z\neq 1$ then equation A requires $x+y=0$ and equation B requires $x+y=1$. It is not possible for both of these to hold. Hence, there are no simultaneous solutions for which $z\neq 1$. This proves the result.

4485. (a) Let $u=\ln x$ and $v'=1$, so that $u'=\frac{1}{x}$ and $v=x$. The parts formula gives

$$\begin{aligned}\int \ln x\,dx&=x\ln x-\int 1\,dx\\ &=x\ln x-x+c.\end{aligned}$$

(b) Using the change of base formula,

$$\log_2 x=\frac{\ln x}{\ln 2}.$$

This gives

$$\begin{aligned}&\int \log_2 x\,dx\\ &=\frac{1}{\ln 2}\int \ln x\,dx\\ &=x\frac{\ln x}{\ln 2}-\frac{x}{\ln 2}+c\\ &\equiv x\log_2 x-\frac{x}{\ln 2}+c,\text{ as required.}\end{aligned}$$

4486. From the graph, the range is $(-\infty,0)\cup[k,\infty)$, where k is the y value of the local minimum. For SPs, we set the derivative to zero:

$$\begin{aligned}\frac{e^x(x+2)-e^x}{(x+2)^2}&=0\\ \Longrightarrow e^x(x+2)-e^x&=0\\ \Longrightarrow x&=-1.\end{aligned}$$

This gives $k=1/e$. So, the range of the function is $(-\infty,0)\cup[1/e,\infty)$.

4487. (a) This is false. A counterexample is $\mathrm{g}(x)=x$ and $\mathrm{f}(x)=x^2+1$. The line $y=x$ has an x intercept, but $y=\mathrm{fg}(x)=x^2+1$ has none.

(b) This is true. Differentiating by the chain rule,

$$\frac{dy}{dx}=\mathrm{f}'(\mathrm{g}(x))\,\mathrm{g}'(x).$$

Wherever $\mathrm{g}'(x)$ is zero, $\mathrm{f}'(\mathrm{g}(x))\,\mathrm{g}'(x)$ is also zero, giving a stationary point of $y=\mathrm{fg}(x)$.

(c) This is false. A counterexample is $\mathrm{g}(x)=x^3$ and $\mathrm{f}(x)=x^2$. The cubic $y=\mathrm{g}(x)=x^3$ has a point of inflection at the origin, but $y=\mathrm{fg}(x)=x^6$ doesn't. Its second derivative is zero, but doesn't change sign.

4488. Integrating by inspection,

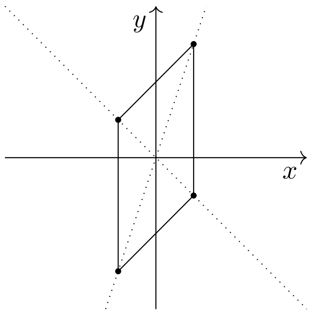
$$\begin{aligned}\int_0^k 2xe^{x^2}+1\,dx&=e\\ \Longrightarrow \Big[e^{x^2}+x\Big]_0^k&=e\\ \Longrightarrow e^{k^2}+k-1&=e\\ \Longrightarrow e^{k^2}-e+k-1&=0.\end{aligned}$$

This is not analytically solvable. The Newton-Raphson iteration is

$$k_{n+1}=k_n-\frac{e^{k_n^2}-e+k_n-1}{2k_ne^{k_n^2}+1}.$$

Running this with $k_0=0$, we get $k_1=e$, then $k_n\to 1$. We can verify that this satisfies the equation exactly. And, since the area function $\mathrm{A}(x)=e^{x^2}+x$ is increasing everywhere, this is the only possible value of k.

4489. Consider the lines $3x - y = 0$ and $x + y = 0$. At any points not on these lines, the algebra is linear. So, the locus is formed of line segments. Setting $3x - y = 0$ gives $(1/2, -1/2)$ and $(-1/2, 1/2)$. Setting $x + y = 0$ gives $(1/2, 3/2)$ and $(-1/2, -3/2)$. These are the vertices of a parallelogram:

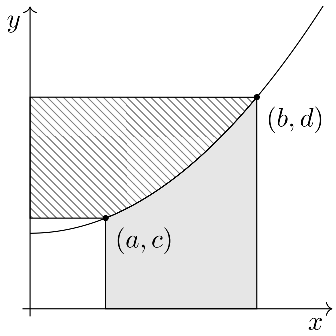


4490. Consider the graph $y = f(x)$.

Since $f^{-1}(y) = x$, we can rewrite the result as

$$\int_a^b y\,dx + \int_c^d x\,dy = bd - ac.$$

Consider this graphically:



The solid shading gives the first integral; the hatched shading gives the second integral. The sum of the two is the total shaded area, which may be calculated as the area bd of the larger rectangle (with a vertex at the origin) minus the area ac of the smaller rectangle. □

4491. The only way the triangles don't overlap is if, taken in order around the circumference, A, B, C and D, E, F form distinct groups. The exact positions of the points aren't relevant, only their order is.

Consider the possibility space as an alphabetical list of 6! orders of the points. For successful orders, there are 6 possible locations for the $\{A, B, C\}$ group. Once this is chosen, there are 3! orders of the $\{A, B, C\}$ group and 3! orders of the $\{D, E, F\}$ group. So, the probability is

$$p = \frac{6 \times 3! \times 3!}{6!} = \tfrac{3}{10}.$$

4492. Let $z = f(x)$ and $y = g(x)$. The first derivative of $f(x)$ with respect to $g(x)$ is

$$\frac{dz}{dy} = \frac{\frac{dz}{dx}}{\frac{dy}{dx}} = \frac{f'(x)}{g'(x)}.$$

Call the above u. The second derivative is

$$\frac{du}{dy} = \frac{\frac{du}{dx}}{\frac{dy}{dx}}.$$

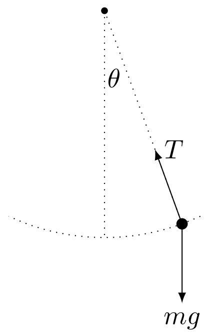
By the quotient rule, the numerator is

$$\frac{du}{dx} = \frac{f''(x)\,g'(x) - f'(x)\,g''(x)}{(g'(x))^2}.$$

Dividing this by $\frac{dy}{dx} = g'(x)$, the second derivative of $f(x)$ with respect to $g(x)$ is

$$\frac{f''(x)\,g'(x) - f'(x)\,g''(x)}{(g'(x))^3}, \text{ as required.}$$

4493. The force diagram is as follows:



Let x be the arc length/position, taken positively from the equilibrium position. With θ in radians, $x = l\theta$. Differentiating twice with respect to t,

$$\frac{d^2x}{dt^2} = l\frac{d^2\theta}{dt^2}.$$

Resolving in the tangential direction,

$$-mg\sin\theta = m\frac{d^2x}{dt^2} = ml\frac{d^2\theta}{dt^2}$$
$$\implies \frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0.$$

For small angles θ in radians, we can approximate $\sin\theta \approx \theta$. This gives

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta \approx 0.$$

— Nota Bene —

The DE above is the equation for *simple harmonic motion*. Its solution curves are sinusoids. Hence, for small oscillations around vertical, a pendulum swings in an approximation (arbitrarily good in the small-angle limit) to sinusoidal motion.

4494. We maximise $y \geq 0$ when we maximise y^2. This is quadratic in $x^{-\frac{4}{5}}$. Completing the square, it is

$$\begin{aligned} y^2 &= -x^{-\frac{8}{5}} + x^{-\frac{4}{5}} \\ &\equiv -\left(x^{-\frac{4}{5}} - \tfrac{1}{2}\right)^2 + \tfrac{1}{4}. \end{aligned}$$

So, the maximum value of y^2 is $1/4$, and therefore the maximum value of y is $1/2$.

4495. There are three possibilities: $Y_1 > Y_2$ or $Y_1 < Y_2$ or $Y_1 = Y_2$. The first two are symmetrical. So, we calculate the third. The probability distribution of $B(5, 1/2)$ is

y	0	1	2	3	4	5
$\mathbb{P}(Y = y)$	$\frac{1}{32}$	$\frac{5}{32}$	$\frac{10}{32}$	$\frac{10}{32}$	$\frac{5}{32}$	$\frac{1}{32}$

$\mathbb{P}(Y_1 = Y_2)$ is given by the sum of the squared probabilities, which is

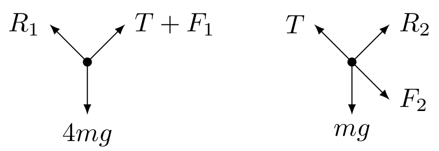
$$2\left({\tfrac{1}{32}}^2 + {\tfrac{5}{32}}^2 + {\tfrac{10}{32}}^2\right) = \tfrac{63}{256}.$$

Subtracting this from 1 gives the probability that Y_1 and Y_2 differ. Halving this gives

$$\begin{aligned} \mathbb{P}(Y_1 > Y_2) &= \tfrac{1}{2}\left(1 - \tfrac{63}{256}\right) \\ &= \tfrac{193}{512}, \text{ as required.} \end{aligned}$$

4496. If the slopes were smooth, then the $4m$ mass would accelerate downwards. Firstly, consider limiting equilibrium. In this case, <u>both</u> frictional forces are maximal, because any motion would require both blocks to move.

The force diagrams are



The reactions are $R_1 = 2\sqrt{2}mg$ and $R_2 = \frac{\sqrt{2}}{2}mg$. So, the maximal frictional forces are $2\sqrt{2}\mu mg$ and $\sqrt{2}\mu mg$ respectively. In the direction of the string, the total component of weight is

$$4mg \cdot \tfrac{\sqrt{2}}{2} - mg \cdot \tfrac{\sqrt{2}}{2} = \tfrac{3\sqrt{2}}{2}mg.$$

The total frictional force is

$$\sqrt{2}\mu mg + 2\sqrt{2}\mu mg = 3\sqrt{2}\mu mg.$$

Equating these,

$$\begin{aligned} \tfrac{3\sqrt{2}}{2}mg &= 3\sqrt{2}\mu mg \\ \Longrightarrow \mu &= \tfrac{1}{2}. \end{aligned}$$

The system is in equilibrium, so $\mu \in [1/2, \infty)$.

4497. Let θ be the angle of inclination of $y = mx$, so that $\tan\theta = m$. The angle of inclination of $y = \sqrt{8}x$ is therefore 2θ. This gives $\sqrt{8} = \tan 2\theta$. Using a double-angle formula,

$$\begin{aligned} \sqrt{8} &= \frac{2\tan\theta}{1 - \tan^2\theta} \\ &= \frac{2m}{1 - m^2}. \end{aligned}$$

This is a quadratic in m:

$$\begin{aligned} \sqrt{8}(1 - m^2) &= 2m \\ \Longrightarrow m &= -\sqrt{2}, \tfrac{\sqrt{2}}{2}. \end{aligned}$$

We reject the former, as the line is in the first quadrant. So, $m = \sqrt{2}/2$.

4498. The problem gives the values p, q and the values u_p, u_q. Using the ordinal formula of a GP, we have (in the problem) two equations:

$$\begin{aligned} u_p &= ar^{p-1}, \\ u_q &= ar^{q-1}. \end{aligned}$$

Dividing one by the other gives

$$r^{p-q} = \frac{u_p}{u_q}.$$

The only way this can generate multiple solutions (thus leaving uncertainty in u_n) is if $p - q$ is even. Hence, the relationship between p and q is that they must have the same parity, i.e. both must be even or both be odd.

4499. Squaring the variable equations,

$$\begin{aligned} s^2 &= x^2 + 2xy + y^2, \\ t^2 &= x^2 - 2xy + y^2. \end{aligned}$$

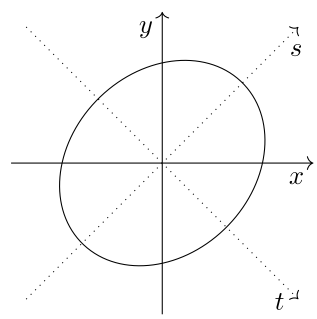
Adding and subtracting these,

$$\begin{aligned} 2x^2 + 2y^2 &= s^2 + t^2, \\ -xy &= \tfrac{1}{4}\left(t^2 - s^2\right). \end{aligned}$$

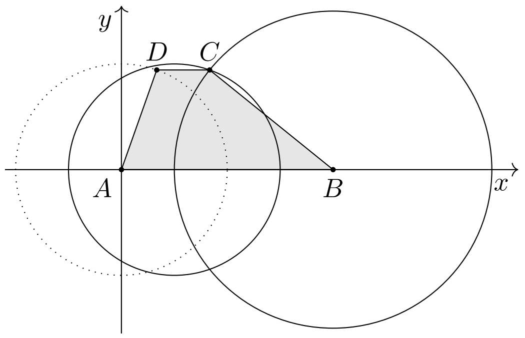
So, the original equation is

$$\begin{aligned} s^2 + t^2 + \tfrac{1}{4}\left(t^2 - s^2\right) &= 1 \\ \Longrightarrow 3s^2 + 5t^2 &= 4. \end{aligned}$$

This is the equation of an ellipse in the (s, t) plane. The value of s is constant along $y = -x$ and the value of t is constant along $y = x$, so the s and t axes are angled at 45° to the x and y axes:



4500. The construction is as follows:



The construction produces a trapezium with edge lengths, not in order, $\{1, 2, 3, 4\}$.

——— End of 45th Hundred ———